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# Measuring bilateral trade in terms of value added

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## Abstract

The increase in the fragmentation of production across countries and the subsequent growth in the trade of intermediate products have raised concerns about the suitability of conventional trade statistics to understand the economic consequences of trade. Several authors have attempted to disentangle value added content of trade. This technical report proposes a novel framework that enables to: (1) fully decompose the factor content of bilateral trade measured at the border; and (2) account for the role of the different countries and industries participating in the global value chain. Furthermore, because of the country and industry detail of this approach, it provides a new extension of the standard value added to exports ratio, and also reconcile the "sink-based" and "source-based" methods commonly used to report the value added in trade.

# 1 Background and policy context

International trade statistics are expected to show a clear picture of the economic implications of international trade in the current context of globalisation, with growing lengths of supply chains due to fragmentation of production. However, the increasing trade of intermediate goods makes the interpretation of trade statistics problematic in the light of key policy issues. Questions that have become difficult to answer include: how much value is actually added by different countries and industries to certain exports, to what extent do the exports of a country/industry rely on inputs from others, or how big are bilateral trade imbalances when measured in terms of value added (VA)?

Conventional trade statistics may give a distorted picture of the relevance of trade for economic growth or improving income inequality. This is because trade flows are measured in gross terms and the value of products that cross borders several times for further processing is counted multiple times (OECD and WTO, 2012). This should be taken into account when interpreting indicators such as the exports to GDP ratio or the bilateral trade balance. This report introduces a novel accounting method that allows a better understanding of the role of the different countries and industries participating in the global value chain.

In order to draw a more accurate picture of the economic implications of trade flows, the different VA components of exports should be reported along with conventional trade statistics. Recent advances in the construction of multiregional input-output tables (MRIO) tables have made this possible. Measuring trade in terms of VA has become crucial for the correct assessment of trade policy and inter-dependencies among countries.

After more than half a century of trade liberalisation, nominal tariffs on manufactured goods in developed and developing economies are lower than ever before. But the implication is not so clear in a world of global value chains. Tariffs and other protection measures at the border are cumulative when intermediate inputs are traded across borders multiple times (OECD, 2015). In this regard, there is a growing concern on the implications of overlapping tariffs. Adding a tariff on final exports to tariffs on inputs (including all tariffs on inputs in the upstream part of the value chain) can lead to high average ad valorem tariffs. This can discourage firms to invest at home and can encourage them to use production facilities, technologies and jobs abroad. This issue might well be analysed using this new accounting method.

Another hot topic in trade policy is the role of services trade. As indicated by (Rueda-Cantuche and Sousa, 2016), services (such as design, engineering or software) embodied in exported goods (denoted as mode 5 services) have become an important part of manufacturing exports and are bound to grow further in the future. This holds in particular for industries such as the automobile industry or electronics (Apple's products are a typical example of this). Also this topic can easily be explored with this accounting framework.

Bilateral trade imbalances seem currently to be a major concern for countries such as the United States and China. However, standard trade statistics hide some critical aspects of trade relationships. Examples are the role of intermediates imports to support exports (import content of exports or vertical specialisation) or the actual size of bilateral trade imbalances when measured in VA terms. Our accounting approach can shed some light to these issues allowing comparisons of bilateral trade balances in both gross and VA terms and from different perspectives, i.e. from the exporter's and from the ultimate consumer's view (Nagengast and Stehrer, 2016). It can also be used to analyse the foreign factor content of exports at country and industry level.

The interest in global value chain analyses goes beyond academia. Also policy makers are increasingly aware of the necessity of complementing existing traditional trade statistics with new indicators better tuned to reflect the position of different countries and industries in the global value chain. This has been reflected in speeches by, for example, the European Commission's President Juncker (State of the Union, 14 September 2016), the OECD Secretary-General Gurría (G20 Trade Ministers Meeting, 6 October 2015) and the former WTO Director-General Lamy (launch event of "Trade patterns and Global Value Chains in East Asia", 6 June 2011).

These policy needs are evolving hand in hand with new developments in terms of databases and analytical tools. In order to better account for the internationalisation and fragmentation of production, new datasets are being developed such as the OECD TiVA database (http://oe.cd/tiva), the Eurostat's FIGARO[1] database and the European Commission funded WIOD database (http://www.wiod.org), among others. Along with the development of new databases, the methodology to correctly disentangle the VA components of international trade has been somewhat drifting around during the last decade with notable contributions such as (Foster-McGregor and Stehrer, 2013; Johnson and Noguera, 2012; Koopman et al., 2014; Los et al., 2015, 2016; Nagengast and Stehrer, 2016; Timmer et al., 2014; Wang et al., 2013). However, as far as the authors know, none of the suggested approaches has been able to: (1) fully decompose the factor content of bilateral trade measured at the border; (2) account for the role of the different countries and industries participating in the global value chain. We therefore propose in this report a new approach to address those two uncovered aspects in the literature.

The remainder of the report is structured as follows. Section 2 introduces the subject and the main contributions of the report. Section 3 presents the decomposition framework. Section 4 discusses the links of the decomposition with the existing measures of the factor content of trade, trade in VA and double counting and Section 5 concludes.

[1] http://ec.europa.eu/eurostat/web/economic-globalisation/globalisation-macroeconomic-statistics/multi-country-supply-use-and-input-output-tables/figaro

# 2 Introduction

As pointed out by (Grossman and Rossi-Hansberg, 2007), the measurement of trade in terms of gross (i.e. intermediate plus final) imports and exports was appropriate when trade flows comprised mostly final goods. But gross trade is an inadequate indicator to understand the consequences of trade in today's world with global supply chains where countries are internationally integrated and with increasing trade on intermediate goods. To that end, it is necessary to know the sources of the value added (VA) embodied in traded goods (i.e. the factor content of trade) and their ultimate destination (Johnson and Noguera, 2012). However, national accounts and trade statistics fail to provide that information, since they just report data on bilateral gross flows of goods and services across borders.

To disentangle the factor content of trade and the fragmentation of global value chains the following measures have been used in the literature (Amador and Cabral, 2016). (*i*) The VA in trade (VAiT), defined as the VA embodied in the gross exports of a country measured at the border. Two components can be distinguished, the domestic and the foreign VAiT (DVAiT and FVAiT). (*ii*) The trade in VA (Trefler and Zhu, 2010) or VA exports (Johnson and Noguera, 2012), which is the VA of one country absorbed by the final demand of other countries. (Johnson and Noguera, 2012) suggest the use of the ratio of the TiVA to gross exports ("VAX ratio") as a summary measure of production fragmentation. (*iii*) The double counted (DC) terms, which are the VA contained in the intermediate goods that cross international borders more than once (Johnson, 2014; Koopman et al., 2014). A distinction can be made between the domestic and foreign DC terms (DDC and FDC).

The recent availability of global multi-regional input-output (MRIO) databases (Tukker and Dietzenbacher, 2013) has been crucial for developing and applying the accounting frameworks above for the decomposition of countries' gross trade (Arto et al., 2015; Dietzenbacher et al., 2013; Foster-McGregor and Stehrer, 2013; Johnson, 2018; Johnson and Noguera, 2012; Koopman et al., 2014; Los et al., 2015, 2016; Nagengast and Stehrer, 2016 and Wang et al., 2013). However, until now, none of the suggested approaches has been able to: (1) fully decompose the factor content of bilateral trade measured at the border; (2) account for the role of the different countries and industries participating in the global value chain.

The method suggested by (Koopman et al., 2014 - KWW, hereafter) is the most popular approach to decompose the factor content of gross exports. These authors provide an innovative unified mathematical framework based on the classical "demand-driven" input-output model (Leontief, 1936) to completely decompose the gross exports into its various components, covering the main measures of the factor content of trade (see expression (36) in KWW). They also show how existing measures of trade in value added and vertical specialisation (Daudin et al., 2011; Hummels et al., 2001) can be derived within their framework. However, their results suffer from three shortcomings.

First, as KWW acknowledge (p. 485), their accounting equation for decomposing the factor content of exports only holds for a country's total exports to all other countries (e.g. total exports of the US) and cannot be used to decompose bilateral exports at the industry level (e.g. US exports of electronics to Mexico). KWW can thus be used to assess the US VA content in its total exports that is ultimately absorbed by the final demand in China. It does not reveal, however, which part of this VA ultimately ends up in China's final demand through the US exports to Mexico. In this sense, the KWW approach cannot be used to fully track the transfers of VA through international trade. This is a major shortcoming of their approach, since the information on the trade partner that enables the transfer of VA to final destinations is essential for the analysis of the economic impact of trade policies. Goods and services are always traded between two specific countries or regions, and trade policy instruments are usually designed to affect bilateral trade flows. Thus, it is natural to also decompose the content of trade at the same scale. In addition, the decomposition method here presented allows extending Johnson and Noguera's

(2012) well-known VAX ratios to the bilateral exports at industry level. Although it is true that (Wang et al., 2013) already extended the KWW framework and considered the factor content of trade at the bilateral and industry level, their method also suffers from the following two caveats, as in KWW.

Second, we show that KWW do not separate foreign VA and DC terms properly, thus leading to different vertical specialisation results when using the foreign value added content of exports. That is, part of the foreign VA in exports is computed as DC and vice versa.

Third, the expressions proposed by KWW and WWZ include gross exports as dependent and independent variables (i.e. gross exports are not only on the left hand side of the equations but also on the right hand side). As a consequence, some applications are not possible. For example, the input-output framework has been widely used to find the drivers of the growth in a certain dependent variable (such as gross exports).[2] A typical question then is: How much of the increase in US exports is due to the growth in Chinese household consumption or changes in the Japanese production structure? The answers cannot be based on the KWW and WWZ framework because of the endogeneity problem in their approach.

Finally, the accounting approach presented in this report can be used to report the VA components of bilateral exports from different perspectives such as that of the ultimate consumer's ("sink-based") and of the exporter's ("source-based") approaches (Nagengast and Stehrer, 2016).[3]

This report proposes an alternative mathematical framework that overcomes these shortcomings. The approach is simple and lies on the foundations of input-output economics and basic matrix algebra. This framework allows decomposing the gross exports of a country, measured at the border, into a single expression. It covers the domestic and foreign VAiT, the TiVA, and the double counting of domestic and foreign VA. It distinguishes the country and industry in which the VA is generated, the exporting country and industry, the importing country and industry, the country and industry producing the final goods and the country whose final demand is driving the exports. Because of the country and industry detail, this method is also able to provide a new extension of the standard VAX ratio.

[2] Early contributions were by (Wolff, 1985) and (Feldman et al., 1987). See (Arto and Dietzenbacher, 2014) for a recent contribution disentangling the annual growth in global greenhouse gas emissions, see (Miller and Blair, 2009) for an overview.

[3] The source-based method refers to the VA from the perspective of the country of production and the sink-based method to VA form the perspective of the country of final absorption.

# 3 Method for the decomposition of gross exports in terms of value added

This section presents the framework for decomposing the gross exports in different VA components. Sections 3.1 and 3.2 present the MRIO model that will be the starting point for the decomposition in section 3.3.

## 3.1 The MRIO model

Following the recent literature on the decomposition of trade in terms of VA, we adopt an MRIO approach to decompose the gross exports measured at the border into the different VA components (Amador and Cabral, 2016). The starting point for the construction of the model is a world MRIO table (see Figure 1), with *m* industries in each of *n* countries. This table describes (in monetary terms) the flows of goods and services between all industries in the world, and the goods and services delivered to final users.

**Figure 1** The World Multi-regional Input-Output Tables

| | Intermediate use | | | | | Final use | | | | | Gross |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | in 1 | ⋯ | in *s* | ⋯ | in *n* | in 1 | ⋯ | in *s* | ⋯ | in *n* | outputs |
| Country 1 | $\mathbf{Z}^{11}$ | ⋯ | $\mathbf{Z}^{1s}$ | ⋯ | $\mathbf{Z}^{1n}$ | $\mathbf{y}^{11}$ | ⋯ | $\mathbf{y}^{1s}$ | ⋯ | $\mathbf{y}^{1n}$ | $\mathbf{x}^{1}$ |
| ⋮ | ⋮ | ⋱ | ⋮ | ⋰ | ⋮ | ⋮ | ⋱ | ⋮ | ⋰ | ⋮ | ⋮ |
| Country *r* | $\mathbf{Z}^{r1}$ | ⋯ | $\mathbf{Z}^{rs}$ | ⋯ | $\mathbf{Z}^{rn}$ | $\mathbf{y}^{r1}$ | ⋯ | $\mathbf{y}^{rs}$ | ⋯ | $\mathbf{y}^{rn}$ | $\mathbf{x}^{r}$ |
| ⋮ | ⋮ | ⋰ | ⋮ | ⋱ | ⋮ | ⋮ | ⋰ | ⋮ | ⋱ | ⋮ | ⋮ |
| Country *n* | $\mathbf{Z}^{n1}$ | ⋯ | $\mathbf{Z}^{ns}$ | ⋯ | $\mathbf{Z}^{nn}$ | $\mathbf{y}^{n1}$ | ⋯ | $\mathbf{y}^{ns}$ | ⋯ | $\mathbf{y}^{nn}$ | $\mathbf{x}^{n}$ |
| Value added | $(\mathbf{w}^{1})'$ | ⋯ | $(\mathbf{w}^{s})'$ | ⋯ | $(\mathbf{w}^{n})'$ | | | | | | |
| Total inputs | $(\mathbf{x}^{1})'$ | ⋯ | $(\mathbf{x}^{s})'$ | ⋯ | $(\mathbf{x}^{n})'$ | | | | | | |

The element $\mathbf{Z}^{rs}$ of the MRIO[4] table is the $m \times m$ matrix of intermediate deliveries from country *r* to country *s*, and its element $z_{ij}^{rs}$ denotes the sales of industry *i* in country *r* to industry *j* in country *s*; $\mathbf{y}^{rs}$ is an $m \times 1$ column vector with aggregated demands by final users (i.e. household consumption, private investments, and government expenditures) and its element $y_i^{rs}$ indicates the final demand in country *s* for goods produced by industry *i* in country *r*. Accordingly, the total exports from country *r* to country *s* (≠ *r*) can therefore be defined as

$$\mathbf{e}^{rs} = \mathbf{Z}^{rs}\mathbf{u}_{(m)} + \mathbf{y}^{rs}, r \neq s \tag{1}$$

where the element $e_i^{rs}$ of the $m \times 1$ vector $\mathbf{e}^{rs}$ represents the total exports of industry *i* in country *r* to (any destination in) country *s*, and $\mathbf{u}_{(m)}$ is the $m \times 1$ summation vector with ones. The $m \times 1$ vector $\mathbf{x}^r$ in Figure 1 gives the gross outputs (or total production) by industry in country *r* and $(\mathbf{w}^s)'$ is the $1 \times m$ vector with the VA by industry in country *s*.

The MRIO table distinguishes four components: the bloc of intermediate deliveries represented by the $nm \times nm$ matrix $\mathbf{Z}$, the bloc of final demands represented by the $nm \times n$ matrix $\mathbf{Y}$, the $nm \times 1$ vector $\mathbf{x}$ of total outputs, and the $nm \times 1$ vector $\mathbf{w}$ of values added. In partitioned form, that is,

[4] Bold-faced lower-case letters are used to indicate vectors, bold-faced capital letters indicate matrices, italic lower-case letters indicate scalars (including elements of a vector or matrix). Subscripts indicate industries and superscripts indicate countries. Vectors are columns by definition, row vectors are obtained by transposition, denoted by a prime (e.g. $\mathbf{x}'$). Diagonal matrices are denoted by $\langle\cdot\rangle$ (e.g. $\langle\mathbf{x}\rangle$ or $\langle\mathbf{Ab}\rangle$ if $\mathbf{x} = \mathbf{Ab}$). Multiple summations like $\sum_{r=1}^{n}\sum_{s=1}^{n}\mathbf{y}^{rs}$ are abbreviated as $\sum_{r,s}\mathbf{y}^{rs}$.

$$\mathbf{Z}=\begin{bmatrix}\mathbf{Z}^{11} & \cdots & \mathbf{Z}^{1s} & \cdots & \mathbf{Z}^{1n}\\ \vdots & \ddots & \vdots & \iddots & \vdots\\ \mathbf{Z}^{r1} & \cdots & \mathbf{Z}^{rs} & \cdots & \mathbf{Z}^{rn}\\ \vdots & \iddots & \vdots & \ddots & \vdots\\ \mathbf{Z}^{n1} & \cdots & \mathbf{Z}^{ns} & \cdots & \mathbf{Z}^{nn}\end{bmatrix},\mathbf{Y}=\begin{bmatrix}\mathbf{y}^{11} & \cdots & \mathbf{y}^{1s} & \cdots & \mathbf{y}^{1n}\\ \vdots & \ddots & \vdots & \iddots & \vdots\\ \mathbf{y}^{r1} & \cdots & \mathbf{y}^{rs} & \cdots & \mathbf{y}^{rn}\\ \vdots & \iddots & \vdots & \ddots & \vdots\\ \mathbf{y}^{n1} & \cdots & \mathbf{y}^{ns} & \cdots & \mathbf{y}^{nn}\end{bmatrix}$$

$$\mathbf{x}=\begin{pmatrix}\mathbf{x}^{1}\\ \vdots\\ \mathbf{x}^{r}\\ \vdots\\ \mathbf{x}^{n}\end{pmatrix},\mathbf{w}=\begin{pmatrix}\mathbf{w}^{1}\\ \vdots\\ \mathbf{w}^{r}\\ \vdots\\ \mathbf{w}^{n}\end{pmatrix}$$

The relation between $\mathbf{x}$, $\mathbf{Z}$ and $\mathbf{Y}$ is defined by the accounting equation $\mathbf{x}=\mathbf{Zu}_{(nm)}+\mathbf{Yu}_{(n)}$. Taking the MRIO table as starting point, the input coefficients matrix is defined as $\mathbf{A}=\mathbf{Z}\langle\mathbf{x}\rangle^{-1}$. In partitioned form, we have

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}^{11} & \cdots & \mathbf{A}^{1s} & \cdots & \mathbf{A}^{1n}\\ \vdots & \ddots & \vdots & \iddots & \vdots\\ \mathbf{A}^{r1} & \cdots & \mathbf{A}^{rs} & \cdots & \mathbf{A}^{rn}\\ \vdots & \iddots & \vdots & \ddots & \vdots\\ \mathbf{A}^{n1} & \cdots & \mathbf{A}^{ns} & \cdots & \mathbf{A}^{nn}\end{bmatrix}$$

where the element $a_{ij}^{rs}$ of $\mathbf{A}^{rs}$ represents the intermediate inputs from industry *i* in country *r* required by industry *j* in country *s* to produce one unit of its output.

The matrix of intermediate deliveries can thus be expressed as $\mathbf{Z}=\mathbf{A}\langle\mathbf{x}\rangle$ and the accounting equation can now be written as the standard input-output equation: $\mathbf{x}=\mathbf{Ax}+\mathbf{Yu}_{(n)}$. For an arbitrary final demand $\mathbf{Y}$, the solution to the model is given by $\mathbf{x}=\mathbf{BYu}_{(n)}$, where $\mathbf{B}$ is the Leontief inverse. That is,

$$\mathbf{B}\equiv(\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}\mathbf{B}^{11} & \cdots & \mathbf{B}^{1s} & \cdots & \mathbf{B}^{1n}\\ \vdots & \ddots & \vdots & \iddots & \vdots\\ \mathbf{B}^{r1} & \cdots & \mathbf{B}^{rs} & \cdots & \mathbf{B}^{rn}\\ \vdots & \iddots & \vdots & \ddots & \vdots\\ \mathbf{B}^{n1} & \cdots & \mathbf{B}^{ns} & \cdots & \mathbf{B}^{nn}\end{bmatrix}$$

Matrix $\mathbf{B}$ contains the output multipliers and the element $b_{ij}^{rs}$ represents the total (direct and indirect) output of industry *i* in country *r* that is required to satisfy one unit of final demand for the goods produced by industry *j* in country *s*.

From $\mathbf{Z}=\mathbf{A}\langle\mathbf{x}\rangle$ and $\mathbf{x}=\mathbf{BYu}_{(n)}$ it follows that the matrix of intermediate deliveries can be expressed as $\mathbf{Z}=\mathbf{A}\langle\mathbf{x}\rangle=\mathbf{A}\langle\mathbf{BYu}_{(n)}\rangle$, and the intermediate exports of country *r* to country *s* as

(2) $$\mathbf{Z}^{rs}=\sum_{p,q}\mathbf{A}^{rs}\langle\mathbf{B}^{sp}\mathbf{y}^{pq}\rangle, r\neq s$$

The VA coefficients are given by the $1 \times (nm)$ vector $\mathbf{v}' = \mathbf{w}'\langle\mathbf{x}\rangle^{-1}$, or $(\mathbf{v}^r)' = (\mathbf{w}^r)'\langle\mathbf{x}^r\rangle^{-1}$ for country *r*. The element $v_j^r$ gives the VA in industry *j* in country *r* per unit of its output. Hence, the VA generated worldwide in the production of the goods and services in order to satisfy total final demand $\mathbf{Y}$ is given by

(3) $$\mathbf{w}'\mathbf{u}_{(nm)} = \mathbf{v}'\mathbf{x} = \mathbf{v}'\mathbf{B}\mathbf{Y}\mathbf{u}_{(n)}$$

For country *r*, the VA generated in the production of the goods and services in order to satisfy total final demand is given by

(4) $$(\mathbf{w}^r)'\mathbf{u}_{(m)} = \textstyle\sum_{s,t}(\mathbf{v}^r)'\mathbf{B}^{rs}\mathbf{y}^{st}$$

The $1 \times m$ vector $(\mathbf{v}^r)'\mathbf{B}^{rs}$ represents the VA multipliers. Its *j*th element is $\sum_i v_i^r b_{ij}^{rs}$ and $v_i^r b_{ij}^{rs}$ gives the VA generated in industry *i* in country *r* (e.g. the equipment industry in India) due to one unit of final demand (e.g. by US households) for goods produced by industry *j* in country *s* (e.g. Japanese cars). This VA multiplier $v_i^r b_{ij}^{rs}$ also includes the feedback effects. These would include, for example the exports of mining tools produced by the equipment industry in India to Australia, which are then used by Australia to produce the coal that is further used by the Indian steel industry to produce intermediate exports that end up in one unit of final demand (e.g. by US households) for Japanese cars.

It should be noted that ultimately the economic value of any product (e.g. Japanese flowers) is the sum of the values that have been added in each stage of the supply chain (VA in India when producing the fertilizers, VA in Korea when producing the garden mould, VA in Japan when cultivating the flowers, etcetera). This can be represented mathematically as follows, $\sum_{r,i} v_i^r b_{ij}^{rs} y_j^{st} = y_j^{st}$, or $\sum_{r,i} v_i^r b_{ij}^{rs} z_{jk}^{st} = z_{jk}^{st}$, or $\sum_{r,i} v_i^r b_{ij}^{rs} = 1$. (The proof is given in Appendix 1). This holds irrespective of the destination: flowers for household consumption, flowers as an intermediate for the production of perfume, or flowers that are exported, they all have the same embodied value added. To quote (Samuelson, 1952): "A rose is a rose is a rose".[5]

## 3.2 Two additional matrices

Before proceeding with the decomposition of the gross exports, we first introduce two additional matrices ($\mathbf{L}^{rr}$ and $\mathbf{C}^{(r)}$) that will be used in the decomposition.

First, the single-country or domestic Leontief inverse matrix of country *r* is given by:

(5) $$\mathbf{L}^{rr} = (\mathbf{I} - \mathbf{A}^{rr})^{-1}$$

The element $l_{ij}^{rr}$ of $\mathbf{L}^{rr}$ gives the total (direct and indirect) output in industry *i* in country *r* required to satisfy one unit of final demand of the goods produced by industry *j* in country *r*, through a strictly domestic value chain. The difference between the MRIO multipliers in $\mathbf{B}^{rr}$ and the single-country multipliers in $\mathbf{L}^{rr}$ are the intraregional feedback effects. The elements $b_{ij}^{rr} - l_{ij}^{rr}$ give the output in industry *i* in country *r* required to satisfy one unit of final demand of the goods produced by industry *j* in country *r*, through value chains that have passed through at least one foreign country. The last (or downstream) stage of production is in country *r*, and so is the first (or upstream) stage. The value chain starts in country *r*, leaves the country *r* and returns at least for the last stage of production (but may have returned and left country *r* many more times in between). This can be seen from comparing the extended versions of both multiplier matrices (see

[5] Samuelson (1952) used this phrase when discussing syllogisms. The original is by Gertrude Stein who wrote the sentence in 1913 in her poem "Sacred Emily".

Appendix 2). From the properties of partitioned matrices (also included in Appendix 2) it follows that $\mathbf{B}^{rr}$ can be expressed as:

(6) $$\mathbf{B}^{rr} = ((\mathbf{I} - \mathbf{A}^{rr}) - \sum_{s \neq r, p \neq r} \mathbf{A}^{rs}\mathbf{C}^{(r)sp}\mathbf{A}^{pr})^{-1}$$

The $(n-1)m \times (n-1)m$ matrix $\mathbf{C}^{(r)}$ is a Leontief inverse matrix that is based on $\mathbf{A}^{(r)}$, i.e. the input matrix from which the rows and columns corresponding to country *r* have been eliminated. That is,

$$\mathbf{A}^{(r)} = \begin{bmatrix} \mathbf{A}^{11} & \cdots & \mathbf{A}^{1,r-1} & \mathbf{A}^{1,r+1} & \cdots & \mathbf{A}^{1n} \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ \mathbf{A}^{r-1,1} & \cdots & \mathbf{A}^{r-1,r-1} & \mathbf{A}^{r-1,r+1} & \cdots & \mathbf{A}^{r-1,n} \\ \mathbf{A}^{r+1,1} & \cdots & \mathbf{A}^{r+1,r-1} & \mathbf{A}^{r+1,r+1} & \cdots & \mathbf{A}^{r+1,n} \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}^{n1} & \cdots & \mathbf{A}^{n,r-1} & \mathbf{A}^{n,r+1} & \cdots & \mathbf{A}^{nn} \end{bmatrix}$$

$$\mathbf{C}^{(r)} = (\mathbf{I} - \mathbf{A}^{(r)})^{-1} = \begin{bmatrix} \mathbf{C}^{(r)11} & \cdots & \mathbf{C}^{(r)1,r-1} & \mathbf{C}^{(r)1,r+1} & \cdots & \mathbf{C}^{(r)1n} \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ \mathbf{C}^{(r)r-1,1} & \cdots & \mathbf{C}^{(r)r-1,r-1} & \mathbf{C}^{(r)r-1,r+1} & \cdots & \mathbf{C}^{(r)r-1,n} \\ \mathbf{C}^{(r)r+1,1} & \cdots & \mathbf{C}^{(r)r+1,r-1} & \mathbf{C}^{(r)r+1,r+1} & \cdots & \mathbf{C}^{(r)r+1,n} \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}^{(r)n1} & \cdots & \mathbf{C}^{(r)n,r-1} & \mathbf{C}^{(r)n,r+1} & \cdots & \mathbf{C}^{(r)nn} \end{bmatrix}$$

The matrix $\mathbf{C}^{(r)}$ computes the output multipliers ignoring the trade links between country *r* and the rest of the countries. It gives all strictly foreign value chains (i.e. value chains in the rest of the world that do not pass through country *r*).

## 3.3 Decomposition of gross exports in terms of VA

Now that the MRIO framework has been presented, the gross exports will be decomposed in two steps. First, the gross exports measured at the border are linked to the final demands that ultimately drive these exports. Next, the exports are decomposed in terms of VA.

To link the gross exports to final demands, consider the final demand in country *q* for goods imported from country *p* (i.e. $\mathbf{y}^{pq}$). The gross output that is required worldwide to produce $\mathbf{y}^{pq}$ amounts to

$$\mathbf{x} = \begin{pmatrix} \mathbf{B}^{1p}\mathbf{y}^{pq} \\ \vdots \\ \mathbf{B}^{sp}\mathbf{y}^{pq} \\ \vdots \\ \mathbf{B}^{np}\mathbf{y}^{pq} \end{pmatrix}$$

According to equation (2), $\mathbf{Z}^{rs} = \mathbf{A}^{rs}\langle\mathbf{x}^{s}\rangle = \mathbf{A}^{rs}\langle\mathbf{B}^{sp}\mathbf{y}^{pq}\rangle$. Using equation (1), it now follows that the gross exports of country $r$ to country $s$, as measured at the border, can be split according to the final demand that ultimately drives them. That is,

(7) $$\mathbf{e}^{rs} = \mathbf{Z}^{rs}\mathbf{u}_{(m)} + \mathbf{y}^{rs} = \sum_{p,q} \mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq} + \mathbf{y}^{rs}, r \neq s$$

which shows the total exports of country *r* to *s* in relation to the final demands that drive them as the sum of two components. (*i*) The intermediate exports $\sum_{p,q}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$ from *r* to *s* that are driven by the final goods produced by any country *p* and consumed by any country *q*. (*ii*) The vector with exports of final goods (or final exports) from *r* to *s*.

The second step is the decomposition of VA in gross exports. Taking equation (7) as a starting point, the VA in the bilateral gross exports is decomposed into two main components: the VA in intermediate exports and the VA in final exports. In order to illustrate this decomposition, consider the trade flows associated with the final demand $\mathbf{y}^{pq}$ in country *q* for goods produced by *p*. For example, the vector of intermediate exports from *r* to *s* (that are necessary for final exports from *p* to *q*) is given by $\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$. These are goods produced in country *r* and their value is built up of values that have been added throughout the production chain. The total value that has been added (to these particular exports) by country *t* is $\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$. Note that summing over *t* yields $\sum_t\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq} = \mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$, because $\sum_t(\mathbf{v}^t)'\mathbf{B}^{tr} = \mathbf{u}'_{(m)}$ and thus $\sum_t\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle = \mathbf{I}$ (see Appendix 1).

The exports of final goods ($\mathbf{y}^{pq}$) from *p* to *q* implies exports of intermediates ($\mathbf{A}^{ts}\mathbf{B}^{sp}\mathbf{y}^{pq}$) from *t* to *s*. Each transaction in this export vector is built up of values added by each country, i.e. $\sum_r\langle(\mathbf{v}^r)'\mathbf{B}^{rt}\rangle\mathbf{A}^{ts}\mathbf{B}^{sp}\mathbf{y}^{pq}$. For example, element *i* of the vector $\langle(\mathbf{v}^r)'\mathbf{B}^{rt}\rangle\mathbf{A}^{ts}\mathbf{B}^{sp}\mathbf{y}^{pq}$ gives the total VA (i.e. by all industries) in country *r* that is embodied in the intermediate exports from industry *i* in country *t* to country *s*.[6] It should be noted that these are only the intermediate exports that are due to the export of final goods ($\mathbf{y}^{pq}$) from *p* to *q*. Element *i* can be written as $\sum_{j,h,k} v_j^r b_{ji}^{rt} a_{ik}^{ts} b_{kh}^{sp} y_h^{pq}$. The expression $v_j^r b_{ji}^{rt} a_{ik}^{ts} b_{kh}^{sp} y_h^{pq}$ would represent, for example, the VA of the Canadian oil industry (country *r*, industry *j*) embodied in the intermediate exports of the Brazilian rubber industry (country *t*, industry *i*) to the French tire industry (country *s*, industry *k*) that is ultimately embodied in the Japanese cars (country *p*, industry *h*) sold to US households (country q).

Note that expression $b_{ji}^{rt} a_{ik}^{ts} b_{kh}^{sp} y_h^{pq}$ distinguishes the country and industry in which the VA is generated, the exporting country and industry, the importing country and industry, the country and industry producing the final goods and the country whose final demand is driving the exports. This means that this method goes a step beyond the "source-based" and "sink-based" approach, as it allows showing simultaneously the information on the exporting country, the importing country, the country in which the value is added and the country which absorbs the value added. In this regard, the "source-based" and "sink-based" approaches could be considered special cases of the general decomposition presented in this report. Thus, summing over the different scripts of $b_{ji}^{rt} a_{ik}^{ts} b_{kh}^{sp} y_h^{pq}$ it is possible to report the VA content of exports in many different ways. For example, the vector $\langle(\mathbf{v}^r)'\mathbf{B}^{rt}\rangle\mathbf{A}^{ts}\mathbf{B}^{sp}\mathbf{y}^{pq}$ provides information on the exporting industry, but not on the industry in country $\mathrm{r}$ in which the VA is generated. Alternatively, one could use $\langle\mathbf{v}^r\rangle\mathbf{B}^{rt}\mathbf{A}^{ts}\mathbf{B}^{sp}\mathbf{y}^{pq}$ in which case element $\mathrm{i}$ gives the VA generated in industry $\mathrm{i}$ in country $\mathrm{r}$ that is embodied in all the intermediate exports of country $\mathrm{t}$ that are imported by country $\mathrm{s}$ (and, of course, due to the exports of final goods from country $p$ to country $\mathrm{q}$).

The vector of intermediate exports of country *r* to country *s* is given in equation (7) by $\mathbf{Z}^{rs}\mathbf{u}_{(m)} = \sum_{p,q}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$. The value added decomposition of these intermediate exports implies $\mathbf{Z}^{rs}\mathbf{u}_{(m)} = \sum_{t,p,q}\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$. In the same fashion can the final exports be decomposed according to VA contributions, i.e. $\mathbf{y}^{rs} = \sum_t\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\,\mathbf{y}^{rs}$. This yields

$$(8)\ \ \mathbf{e}^{rs} = \mathbf{Z}^{rs}\mathbf{u}_{(m)} + \mathbf{y}^{rs} = \underbrace{\textstyle\sum_{t,p,q}\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{I} + \underbrace{\textstyle\sum_t\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\,\mathbf{y}^{rs}}_{II}, r \neq s$$

[6] If one would like to distinguish also the receiving industry in country *s* (say industry *k*), element (*i*, *k*) of the matrix $\langle(\mathbf{v}^r)'\mathbf{B}^{rt}\rangle\mathbf{A}^{ts}\langle\mathbf{B}^{sp}\mathbf{y}^{pq}\rangle$ should be considered.

Each of the components $I$ and $II$ can be split into two, distinguishing between $t = r$ and $t \neq r$. This results in two components with domestic value added (*a* and *c*) and two with foreign value added (*b* and *d*),

$$(9) \qquad \mathbf{e}^{rs} = \underbrace{\textstyle\sum_{p,q}\langle(\mathbf{v}^r)'\mathbf{B}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{a} + \underbrace{\textstyle\sum_{p,q,t\neq r}\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{b} + \underbrace{\langle(\mathbf{v}^r)'\mathbf{B}^{rr}\rangle\mathbf{y}^{rs}}_{c} + \underbrace{\textstyle\sum_{t\neq r}\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\,\mathbf{y}^{rs}}_{d}, r \neq s$$

Next, expression *a* is split in order to distinguish between the value chains that are entirely domestic (in expression *ii*, below) and those that have passed at least once through a country other than *r* (expression *i*, below). In the same way, expression *c* (also with domestic value added) is split below into *v* and *vi*.

We would like to make a similar split for expressions *b* and *d* with foreign value added. We first write $\mathbf{B}^{tr} = \sum_{z\neq r}\mathbf{B}^{tz}\mathbf{A}^{zr}\mathbf{L}^{rr}$ (which follows from the properties of the inverse of a partitioned matrix, see Appendix 2). Expression *d* for example then becomes $\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^t)'\mathbf{B}^{tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{y}^{rs}$. The first part $\sum_{z\neq r,t\neq r}(\mathbf{v}^t)'\mathbf{B}^{tz}$ gives VA in the rest of the world (RoW, i.e. anywhere but in country *r*) embodied in products produced in RoW. Next these chains are split into two parts: chains entirely in RoW ($\sum_{z\neq r,t\neq r}(\mathbf{v}^t)'\mathbf{C}^{(r)tz}$, leading to expression *viii* below) and chains (beginning and ending in RoW) that have passed at least once through country *r* ($\sum_{z\neq r,t\neq r}(\mathbf{v}^t)'(\mathbf{B}^{tz} - \mathbf{C}^{(r)tz})$, leading to *vii*). Combining all elements yields

$$(10) \quad \mathbf{e}^{rs} = \underbrace{\textstyle\sum_{p,q}\langle(\mathbf{v}^r)'(\mathbf{B}^{rr} - \mathbf{L}^{rr})\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{i} + \underbrace{\textstyle\sum_{p,q}\langle(\mathbf{v}^r)'\mathbf{L}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{ii}$$

$$+\underbrace{\textstyle\sum_{p,q,z\neq r,t\neq r}\langle(\mathbf{v}^t)'(\mathbf{B}^{tz} - \mathbf{C}^{(r)tz})\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{iii}$$

$$+ \underbrace{\textstyle\sum_{p,q,z\neq r,t\neq r}\langle(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{iv} + \underbrace{\langle(\mathbf{v}^r)'(\mathbf{B}^{rr} - \mathbf{L}^{rr})\rangle\mathbf{y}^{rs}}_{v}$$

$$+ \underbrace{\langle(\mathbf{v}^r)'\mathbf{L}^{rr}\rangle\mathbf{y}^{rs}}_{vi} + \underbrace{\textstyle\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^t)'(\mathbf{B}^{tz} - \mathbf{C}^{(r)tz})\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{y}^{rs}}_{vii}$$

$$+ \underbrace{\textstyle\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{y}^{rs}}_{viii}, r \neq s$$

Expression (10) decomposes the gross exports of country of *r* to country *s* in eight elements that can be linked to the main measures of the factor content of trade and trade in value added (which is covered in the next section). Observe that equation (10), unlike the main expression in KWW, shows no endogeneity issues. That is, none of the components of the decomposition of the gross exports depends on gross exports. All components of the decomposition depend on final demands, which is fully consistent with Leontief's "demand-driven" input-output model.

Furthermore, expression (10) can be re-written in order to show information on the industries involved in the global supply chain. Accordingly, the exports of industry $i$ of country $r$ to country $s$ can be expressed as:

(11)

$$
\begin{aligned}
e_i^{rs} = {} & \underbrace{\sum_{h,j,k}\sum_{p,q} v_h^r (b_{hi}^{rr} - l_{hi}^{rr}) a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}}_{i} + \underbrace{\sum_{h,j,k}\sum_{p,q} v_h^r l_{hi}^{rr} a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}}_{ii} + \\
& \underbrace{\sum_{f,g,h,j,k}\sum_{p,q,z\neq r,t\neq r} v_f^t \left(b_{fg}^{tz} - c_{fg}^{(r)tz}\right) a_{gh}^{zr} l_{hi}^{rr} a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}}_{iii} + \\
& \underbrace{\sum_{f,g,h,j,k}\sum_{p,q,z\neq r,t\neq r} v_f^t c_{fg}^{(r)tz} a_{gh}^{zr} l_{hi}^{rr} a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}}_{iv} + \underbrace{\sum_h v_h^r (b_{hi}^{rr} - l_{hi}^{rr}) y_i^{rs}}_{v} + \underbrace{\sum_h v_h^r l_{hi}^{rr} y_i^{rs}}_{vi} + \\
& \underbrace{\sum_{f,g,h}\sum_{z\neq r,t\neq r} v_f^t \left(b_{fg}^{tz} - c_{fg}^{(r)tz}\right) a_{gh}^{zr} l_{hi}^{rr} y_i^{rs}}_{vii} + \underbrace{\sum_{f,g,h}\sum_{z\neq r,t\neq r} v_f^t c_{fg}^{(r)tz} a_{gh}^{zr} l_{hi}^{rr} y_i^{rs}}_{viii} \quad with\, r \neq s
\end{aligned}
$$

Expression (11) represents the gross exports of industry $i$ of country of $r$ to country $s$ as the sum of a set of elements reporting information on the country and industry in which the VA is generated, the exporting country and industry, the importing country and industry, the country and industry producing the final good, and the country which ultimately absorbs the VA:

- $v_h^r l_{hi}^{rr} a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}$ is the VA generated by industry $h$ of country $r$, that is embodied in the intermediate exports of industry $i$ of country $r$ to industry $j$ of country $s$, and ends up in the demand of country $q$ of final products produced by industry $k$ of country $p$ ($v_h^r (b_{hi}^{rr} - l_{hi}^{rr}) a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}$ is the corresponding DC term);
- $v_f^t c_{fg}^{(r)tz} a_{gh}^{zr} l_{hi}^{rr} a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}$ is the VA generated by industry $f$ of country $t$, that is embodied in the intermediate exports of industry $g$ of country $z$ to industry $h$ of country $r$, that is further embodied in the intermediate exports of industry $i$ of country $r$ to industry $j$ of country $s$, and ends up in the demand of country $q$ of final products produced by industry $k$ of country $p$ ($v_f^t \left(b_{fg}^{tz} - c_{fg}^{(r)tz}\right) a_{gh}^{zr} l_{hi}^{rr} y_i^{rs}$ is the corresponding DC term);
- $v_h^r l_{hi}^{rr} y_i^{rs}$ is the VA generated by industry $h$ of country $r$, that is embodied in the final exports of industry $i$ of country $r$ to country $s$ ($v_h^r (b_{hi}^{rr} - l_{hi}^{rr}) y_i^{rs}$ is the corresponding DC term);
- $v_f^t c_{fg}^{(r)tz} a_{gh}^{zr} l_{hi}^{rr} y_i^{rs}$ is the VA generated by industry $f$ of country $t$, that is embodied in the intermediate exports of industry $g$ of country $z$ to industry $h$ of country $r$, that is embodied in the final exports of industry $i$ of country $r$ to country $s$ ($v_f^t \left(b_{fg}^{tz} - c_{fg}^{(r)tz}\right) a_{gh}^{zr} l_{hi}^{rr} y_i^{rs}$ is the corresponding DC term.

# 4 Link with measures of value added in trade, trade in value added and double counted terms

This section shows the links between the different elements of the decomposition of the gross exports in terms of VA in equation (10) and the key indicators used in the literature. These are: VAiT, TiVA, and DC terms. The links between the indicators are summarised in Figure 2.

## 4.1 Value Added in Trade

The VAiT is defined as the VA embodied in the gross exports of a country measured at the border. We can distinguish two components, the domestic and the foreign VA (DVAiT and FVAiT).

The DVAiT in the exports of country *r* to country *s* is the VA generated in country *r* when producing its exports to *s* (e.g. the VA of all the Indian industries embodied in the Indian exports to Japan). Note, however, that DVAiT must only cover the domestic (e.g. Indian) VA that is derived from the purely domestic part of the supply chains. To see why this is the case, suppose that the Indian steel manufacturing industry exports to a Japanese factory that makes car parts which are exported to the Indian automobile industry that exports also to Japanese households. The gross exports then include Indian steel and Indian cars (which embody some Indian steel). In order to prevent double counting, the VA in the exports of Indian steel should be measured only once. The VA in the exports of Indian cars should therefore neglect the VA by Indian steel manufacturing that is embodied in the Indian cars. Instead it should only measure the VA in, for example, the Indian assembly of the car from different parts. The domestic VA in the exports $\mathbf{e}^{rs}$ of country $r$ to country $s$ is given by

$$(12) \qquad \mathbf{d}^{rs} = \langle(\mathbf{v}^r)'\mathbf{L}^{rr}\rangle\mathbf{e}^{rs} = \textstyle\sum_{p,q}\langle(\mathbf{v}^r)'\mathbf{L}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq} + \langle(\mathbf{v}^r)'\mathbf{L}^{rr}\rangle\mathbf{y}^{rs}$$

where we have used (7). The use of $\mathbf{L}^{rr}$ (instead of $\mathbf{B}^{rr}$) means that we consider only the supply chains that are strictly domestic. Note that equation (12) is the same as the sum of the elements *ii* and *vi* in (10).

The FVAiT in the exports of country *r* to country *s* is defined as the VA generated in countries other than *r* when producing the intermediate imports of *r* that are used by *r* to produce its exports to *s* (see Arto et al., 2015). That is,

$$(13) \qquad \mathbf{f}^{rs} = \textstyle\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{e}^{rs}$$

$$= \textstyle\sum_{p,q,z\neq r,t\neq r}\langle(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq} + \textstyle\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{y}^{rs}$$

which corresponds to the elements *iv* and *viii* of (10). Country *r* produces exports to *s*, for which it requires intermediate imports (reflected by $\mathbf{A}^{zr}\mathbf{L}^{rr}$). The production of these intermediates involves foreign value added. In order to avoid double counting, only the foreign part of the supply chains is taken into account. That is, the FVAiT is calculated using the Leontief inverse of the MRIO model that excludes country *r*. Consider a similar example as before. That is, the automobile industry in India (country *r*) exports cars to households in Japan (country *s*) and for the production it uses car parts from Korea (country *z*). Expression (13) includes the steel from Russia (country *t*) that is needed to make the Korean car parts. Expression (13) does *not* include the Russian iron ores that are used by the Indian steel manufacturing to produce steel for the Korean car parts.

**Figure 2** Decomposition of the total gross exports of country r to country s in terms of VA, and links with VAiT, TiVA, and DC

<table>
<tr><th colspan="4" rowspan="2">Decomposition of gross exports</th><th colspan="3">Domestic</th><th colspan="2">Foreign</th></tr>
<tr><th>VAiT</th><th>TiVA</th><th>DC</th><th>VAiT</th><th>DC</th></tr>
<tr><td rowspan="4">Intermediate exports</td><td rowspan="4">$\underbrace{\sum_{t,p,q}\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{I}$</td><td rowspan="2">$\underbrace{\sum_{p,q}\langle(\mathbf{v}^r)'\mathbf{B}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{a}$</td><td>$\underbrace{\sum_{p,q}\langle(\mathbf{v}^r)'(\mathbf{B}^{rr}-\mathbf{L}^{rr})\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{i}$</td><td></td><td></td><td>X</td><td></td><td></td></tr>
<tr><td>$\underbrace{\sum_{p,q}\langle(\mathbf{v}^r)'\mathbf{L}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{ii}$</td><td>X</td><td>$q \neq r$</td><td></td><td></td><td></td></tr>
<tr><td rowspan="2">$\underbrace{\sum_{p,q,t\neq r}\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{b}$</td><td>$\underbrace{\sum_{p,q,z\neq r,t\neq r}\langle(\mathbf{v}^t)'(\mathbf{B}^{tz}-\mathbf{C}^{(r)tz})\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{iii}$</td><td></td><td></td><td></td><td></td><td>X</td></tr>
<tr><td>$\underbrace{\sum_{p,q,z\neq r,t\neq r}\langle(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{iv}$</td><td></td><td></td><td></td><td>X</td><td></td></tr>
<tr><td rowspan="4">Final exports</td><td rowspan="4">$\underbrace{\sum_{t}\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{y}^{rs}}_{II}$</td><td rowspan="2">$\underbrace{\langle(\mathbf{v}^r)'\mathbf{B}^{rr}\rangle\mathbf{y}^{rs}}_{c}$</td><td>$\underbrace{\langle(\mathbf{v}^r)'(\mathbf{B}^{rr}-\mathbf{L}^{rr})\rangle\mathbf{y}^{rs}}_{v}$</td><td></td><td></td><td>X</td><td></td><td></td></tr>
<tr><td>$\underbrace{\langle(\mathbf{v}^r)'\mathbf{L}^{rr}\rangle\mathbf{y}^{rs}}_{vi}$</td><td>X</td><td>X</td><td></td><td></td><td></td></tr>
<tr><td rowspan="2">$\underbrace{\sum_{t\neq r}\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{y}^{rs}}_{d}$</td><td>$\underbrace{\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^t)'(\mathbf{B}^{tz}-\mathbf{C}^{(r)tz})\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{y}^{rs}}_{vii}$</td><td></td><td></td><td></td><td></td><td>X</td></tr>
<tr><td>$\underbrace{\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{y}^{rs}}_{viii}$</td><td></td><td></td><td></td><td>X</td><td></td></tr>
</table>

This would be double counting because this Russian VA in iron ores is accounted for elsewhere, namely in the foreign VA in the Indian gross exports of steel for Korean car parts production. This double counting would occur if the vector of Indian gross exports would be multiplied with the Leontief inverse of the full global MRIO model (i.e. $\mathbf{B}^{tz}$). The matrix $\mathbf{C}^{(r)tz}$ does not include supply chains that run through India (country *r*) and is the appropriate matrix to be used when calculating the FVAiT.[7] From this example it also follows that DC terms would be captured by the difference between $\mathbf{B}^{tz}$ and $\mathbf{C}^{(r)tz}$.

## 4.2 Trade in Value Added

The second measure is the TiVA, which is defined as the VA of one country (*r*) absorbed by the final demands of all the other countries ($q \neq r$). For country *r*, the TiVA is given by the scalar

$$(14) \qquad \sum_{p,q\neq r}(\mathbf{v}^r)'\mathbf{B}^{rp}\mathbf{y}^{pq}$$

It should be emphasised that TiVA is a scalar whereas the expressions in (10) are all vectors. We will show how TiVA can be obtained from a set of sums of vectors in (10).

Appendix 3 shows that TiVA can be written as $\sum_{s\neq r}\tau^{rs}$, with

$$(15) \qquad \tau^{rs} = \sum_{p,q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq} + (\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{y}^{rs}$$

Note that $\sum_{p,q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$ is the sum of vector (*ii*) in (10) with $q \neq r$ and that $(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{y}^{rs}$ is the sum of vector (*vi*) in (10). TiVA thus consists of two components,

$$(16) \qquad TiVA = \sum_{s\neq r,p,q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq} + \sum_{s\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{y}^{rs}$$

where $\sum_{s\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{y}^{rs}$ gives the VA in country *r* embodied in the deliveries of its final products to foreign consumers. The term $\sum_{s\neq r,p,q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$ looks at all purchases of foreign (i.e. $q \neq r$, for any *p*) consumers, calculates how much foreign production (i.e. $s \neq r$) is required, how much intermediate imports from country *r* are necessary and how much VA is involved in *r*. So, there are two ways for the VA created in country *r* to become absorbed by foreign consumers: through the exports of final products and through the exports of intermediate products via country *s*.

When DVAiT is taken as a scalar (by taking the vector sums), it contains TiVA as a subset. Expression (12) gives the domestic VA in the exports of *r* to *s*. The domestic VA in all exports of *r* is thus obtained by summing expression (12) over $s \neq r$, which almost equals (16). We have

$$DVAiT = \sum_{s\neq r,p,q}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq} + \sum_{s\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{y}^{rs}$$

and

$$(17) \qquad TiVA = DVAiT - \sum_{s\neq r,p}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pr}$$

[7] Arto et al. (2015) use this matrix to compute the foreign VA in the exports of the EU.

The difference between TiVA and DVAiT gives the domestic VA in the exports of intermediates from *r* to *s* that returns home and is ultimately absorbed by the final demands of the exporting country *r*.

Note that $(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sr}\mathbf{y}^{rs}$ is part of DVAiT and TiVA. It gives the VA in the intermediate exports from *r* to any country *s* that returns home and is ultimately embodied in the final exports of *r* to *s*. In (10), this VA in re-exports is included in *ii* (when $p = r, q = s$) but also in *v* (see Appendix 4). This reflects (part of) the double counting of VA that is present in the gross exports. Element *v* in (10) is thus labeled as DC.

Expression (17) can be also used to compute the VA exports ratio ("VAX ratio") at the bilateral and industry level. (Johnson and Noguera, 2012) define the VAX ratio of a country as the VA absorbed by foreign final demands divided by its gross exports. Accordingly, the "VAX ratio" at the bilateral and industry level would be the VA of a country *r* that is embedded in the exports of its industry $i$ to country *s* and that is absorbed abroad (TiVA) divided by the total exports of industry *i* of country *r* to country *s*. That is,

$$VAXratio_i^{rs} = \frac{\sum_{h,j,k,p,q\neq r} v_h^r l_{hi}^{rr} a_{ij}^{rs} b_{jk}^{sp} y_k^{pq} + \sum_h v_h^r l_{hi}^{rr} y_i^{rs}}{e_i^{rs}}$$

The bilateral VAX ratio is obtained as a weighted average, where the weights are the industry shares in the exports from *r* to *s*,

$$VAXratio^{rs} = \sum_i VAXratio_i^{rs} \frac{e_i^{rs}}{\sum_i e_i^{rs}}$$

The VAX ratio as defined in (Johnson and Noguera, 2012) is then obtained by taking a further weighted average, with the exports shares by destination as weights,

$$VAXratio = \sum_{s\neq r} VAXratio^{rs} \frac{\sum_i e_i^{rs}}{\sum_{s\neq r,i} e_i^{rs}} = \frac{TiVA}{\sum_{s\neq r,i} e_i^{rs}}$$

## 4.3 Double counted terms

The double counted (DC) value added is the difference between the gross exports and the VAiT. It refers to the intermediate goods that cross international borders more than once (Johnson, 2014; Koopman et al., 2014). The first step to split the exports into VAiT and DC terms consists of expressing the bilateral exports in terms of VA.

$$\mathbf{e}^{rs} = \sum_t \langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{e}^{rs}$$

This expression can be split into domestic VA content ($\langle(\mathbf{v}^r)'\mathbf{B}^{rr}\rangle\mathbf{e}^{rs}$) and foreign VA content ($\sum_{t\neq r}\langle(\mathbf{v}^t)'\mathbf{B}^{tr}\rangle\mathbf{e}^{rs}$). Both the domestic and the foreign VA content can be further split into "pure" VA and DC terms,

$$\langle(\mathbf{v}^r)'\mathbf{B}^{rr}\rangle\mathbf{e}^{rs} = \underbrace{\langle(\mathbf{v}^r)'\mathbf{L}^{rr}\rangle\mathbf{e}^{rs}}_{DVAiT} + \underbrace{\langle(\mathbf{v}^r)'(\mathbf{B}^{rr} - \mathbf{L}^{rr})\rangle\mathbf{e}^{rs}}_{DDC}$$

Using $\mathbf{B}^{tr} = \sum_{z\neq r}\mathbf{B}^{tz}\mathbf{A}^{zr}\mathbf{L}^{rr}$ yields for the foreign VA content $\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^{t})'\mathbf{B}^{tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{e}^{rs}$ which is then split into

$$\sum_{t\neq r}\langle(\mathbf{v}^{t})'\mathbf{B}^{tr}\rangle\mathbf{e}^{rs} = \underbrace{\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^{t})'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{e}^{rs}}_{FVAiT} +$$

$$+\underbrace{\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^{t})'(\mathbf{B}^{tz}-\mathbf{C}^{(r)tz})\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{e}^{rs}}_{FDC}$$

Using (7) again yields for the double counting of domestic VA

$$(18) \qquad \underbrace{\langle(\mathbf{v}^{r})'(\mathbf{B}^{rr}-\mathbf{L}^{rr})\rangle\mathbf{e}^{rs}}_{DDC} = \underbrace{\sum_{p,q}\langle(\mathbf{v}^{r})'(\mathbf{B}^{rr}-\mathbf{L}^{rr})\rangle\,\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{i}$$

$$+\ \underbrace{\langle(\mathbf{v}^{r})'(\mathbf{B}^{rr}-\mathbf{L}^{rr})\rangle\mathbf{y}^{rs}}_{v}$$

and for the double counting of foreign VA

$$(19) \qquad \underbrace{\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^{t})'(\mathbf{B}^{tz}-\mathbf{C}^{(r)tz})\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{e}^{rs}}_{FDC}$$

$$=\ \underbrace{\sum_{p,q,z\neq r,t\neq r}\langle(\mathbf{v}^{t})'(\mathbf{B}^{tz}-\mathbf{C}^{(r)tz})\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}}_{iii}$$

$$+\ \underbrace{\sum_{z\neq r,t\neq r}\langle(\mathbf{v}^{t})'(\mathbf{B}^{tz}-\mathbf{C}^{(r)tz})\mathbf{A}^{zr}\mathbf{L}^{rr}\rangle\mathbf{y}^{rs}}_{vii}$$

The double counted terms in expressions (17) and (18) compute the domestic and foreign VA in the exports of *r* to *s* that crosses the border several times. These DC terms are calculated as the difference between the multipliers in the full MRIO model and the multipliers in the restricted model. In (17), the restricted model consists only of country *r* so that $\mathbf{B}^{rr}-\mathbf{L}^{rr}$ is used. In (18), the restricted model consists of all countries except country *r* which implies that $\mathbf{B}^{tz}-\mathbf{C}^{(r)tz}$ is used. The reason is that the vector of gross exports also takes feedback effects into account which implies double counting. For example, Indian exports to Japan include the exports of Indian steel to Japan as well as the exports of mining equipment from India to Australia that is used in Australia to produce the coal that is used by the Indian steel industry to produce its exports to Japan. Using the Leontief inverse of the full model always includes the VA in the trade flows that are generated due to the feedback effects which are thus double counted. In this example that would be the VA created in the Indian mining equipment industry, due to exports of Indian steel to Japan.

## 4.4 Value Added in trade from the "sink" and "source" perspectives

Expression (11) is the general case of the decomposition of the gross exports of industry $i$ of country $r$ to country $s$. The VAiT from the "source-based" and "sink-based" approaches can be easy derived from (11) by just modifying the arguments of the summations.

For example, from the "source" perspective, the foreign VA in country $t \neq r$ that is embodied in the intermediate exports of industry $i$ of country $r$ to country $s$, regardless the country that ultimately absorbs the VA can be calculated as $\sum_{f,g,h,j,k} \sum_{p,q,z\neq r} v_f^t c_{fg}^{(r)tz} a_{gh}^{zr} l_{hi}^{rr} a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}$. From the "sink" perspective, the foreign VA in all country $t \neq r$ that is embodied in the intermediate exports of industry $i$ of country $r$ to country $s$, that ends up in the final demand of country $q$ is denoted $\sum_{f,g,h,j} \sum_{p,z\neq r,t\neq r} v_f^t c_{fg}^{(r)tz} a_{gh}^{zr} l_{hi}^{rr} a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}$. Similar expressions can be derived for final export.

Furthermore, from (11) we can derive an expression for the VA in bilateral trade showing simultaneously the country in which the value added is generated ("source") and the country which absorbs the value added ("sink"). For example, the VA in country $r$ that is embodied in the intermediate exports of industry $i$ of country $r$ to country $s$, that ends up in the final demand of country $q$ is $\sum_{h,j,k} \sum_p v_h^r l_{hi}^{rr} a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}$. The foreign VA in country $t \neq r$ that is embodied in the exports of industry $i$ of country $r$ to country $s$, that ends up in the final demand of country $q$ is $\sum_{f,g,h,j} \sum_{p,z\neq r} v_f^t c_{fg}^{(r)tz} a_{gh}^{zr} l_{hi}^{rr} a_{ij}^{rs} b_{jk}^{sp} y_k^{pq}$.

## 4.5 Comparison with KWW

This section compares the decomposition framework with that of KWW. Unlike the main expression in KWW, equation (10) shows no endogeneity issues. That is, none of the components of the decomposition of the gross exports depends on gross exports. All components of the decomposition depend on final demands, which is fully consistent with Leontief's "demand-driven" input-output model. Another major difference is that the decomposition of FVAiT in (13) differs from the one reported in KWW. We will show that KWW do not properly separate the FVA and FDC components of the gross exports: KWW compute part of the FVA as FDC and vice versa.[8]

In KWW, the "foreign VA in intermediate good exports" (as a scalar) is given by:

$$\sum_{s\neq r,t\neq r} (\mathbf{v}^t)'\mathbf{B}^{tr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{y}^{ss} \qquad (20)$$

There are three points where (20) differs from $\sum_{p,q,z\neq r,t\neq r} (\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$ in (13). First, $\mathbf{y}^{ss}$ (for all $s \neq r$) is used in (20) instead of $\mathbf{y}^{pq}$ (for any $p$ and $q$). (20) thus looks only at the FVA in the intermediate exports of $r$ to $s$ that are related to the final demand $\mathbf{y}^{ss}$ for domestically produced goods in the importing country $s$. (20) seriously underestimates the FVA in intermediate exports in (13).

The second point is that KWW use $\mathbf{L}^{ss}$ in (20), whereas the decomposition in (13) uses $\mathbf{B}^{sp}$. The $\mathbf{B}$ matrices are used because the output in country $s$ is required (i.e. $\mathbf{x}^s = \sum_{p,q} \mathbf{B}^{sp}\mathbf{y}^{pq}$), after which the intermediate imports can be determined (i.e. $\mathbf{A}^{rs}\mathbf{x}^s$). This yields another underestimation because $\mathbf{B}^{sp} \geq 0$ and for $p = s$ even $\mathbf{B}^{ss} \gg \mathbf{L}^{ss}$ holds under mild conditions.

The third point is that KWW use the global multiplier of the MRIO to compute the foreign VA in the intermediate exports absorbed by the direct importer. That is, (20) uses $(\mathbf{v}^t)'\mathbf{B}^{tr}$ whilst we use $(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}$ in (13). Los, Timmer and de Vries (2016, p. 1965) have a problem with the use of $(\mathbf{v}^t)'\mathbf{B}^{tr}$ and state: "while mathematically correct, it does not have a clear interpretation. One might be led to think that the multiplication of an export vector with the Leontief inverse, can be interpreted as the gross output associated with the production of exports. This only holds true however when exports exclusively contain exogenous elements of final demand". As we have indicated earlier, $(\mathbf{v}^t)'\mathbf{B}^{tr}$ may well be used to split the value of a product into the values that have been added in the stages of

[8] (Nagengast and Stehrer, 2016) decompose bilateral trade balances and use the global multiplier to compute the VA in intermediate exports. Similar to KWW, they also mix "pure" VA and DC terms. (This holds for the elements 7, 10, 12, 13, 15, 16, 18, 20, 21, 22, 23, 24, 25 and 26 of their equation (7)).

the supply chain. No matter whether a rose is part of the final demand or of the intermediate exports, "[a] rose is a rose is a rose" (Samuelson, 1952, p. 58).

The problem with KWW is with respect to which parts of the value chain are taken into account. In order to avoid double counting, the FVAiT has to be calculated using the Leontief inverse of the MRIO model that excludes country $r$ (i.e. $\mathbf{C}^{(r)tz}$). By using the Leontief inverse of the full global MRIO model, KWW are computing not only the FVA in intermediate exports absorbed by the direct importer, but also the FDC terms. This can be seen as follows. In deriving equations (18) and (19), we have used $(\mathbf{v}^t)'\mathbf{B}^{tr} = \sum_{z\neq r}(\mathbf{v}^t)'\mathbf{B}^{tz}\mathbf{A}^{zr}\mathbf{L}^{rr}$ and $\mathbf{B}^{tz} = \mathbf{C}^{(r)tz} + (\mathbf{B}^{tz} - \mathbf{C}^{(r)tz})$. The parts with $\mathbf{C}^{(r)tz}$ reflected "pure" foreign VA (i.e. FVA) and those with $(\mathbf{B}^{tz} - \mathbf{C}^{(r)tz})$ double counted foreign VA (i.e. FDC). The inclusion of double counted terms by KWW would yield an overestimation.

Next, we show that part of the FVA in intermediate exports in KWW is included as FDC in (10). Vice versa, part of the FDC in KWW is FVA in intermediate exports in (10).

The FVA in intermediate exports in KWW is given in (20) and can be further split as

$$(21)\quad \sum_{s\neq r,t\neq r}(\mathbf{v}^t)'\mathbf{B}^{tr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{y}^{ss} = \sum_{s\neq r,t\neq r,z\neq r}(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{y}^{ss}$$

$$+ \sum_{s\neq r,t\neq r,z\neq r}(\mathbf{v}^t)'\left(\mathbf{B}^{tz} - \mathbf{C}^{(r)tz}\right)\mathbf{A}^{zr}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{y}^{ss}$$

The first term on the right hand side of (21) represents the part of the foreign VA in the intermediate exports of $r$ to $s$ that is absorbed by any direct importer (i.e. $s$). It is included in our expression for the FVA. It equals part $iv$ of (10) with $p = s$, $q = s$, $s = 1,\ldots,m$, $s \neq r$, and where $\mathbf{L}^{ss}$ is used instead of $\mathbf{B}^{ss}$.[9]

The second term on the right hand side of (21) is part of the FVA in intermediate exports in KWW. In our decomposition this term is a large part of the FDC that is associated to the intermediate exports absorbed by the direct importer. It equals part $iii$ of (10) with $p = s$, $q = s$, $s = 1,\ldots,m$, $s \neq r$, and where $\mathbf{L}^{ss}$ is used instead of $\mathbf{B}^{ss}$. This shows that part of the FVA in KWW is double counted foreign value added (and belongs to FDC).

The expression for the foreign DC terms in KWW ("DC foreign VA") is $\sum_{s\neq r,t\neq r,q}(\mathbf{v}^t)'\mathbf{B}^{tr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{e}^{sq}$, which can be split as follows

$$(22)\quad \sum_{t\neq r,s\neq r,q}(\mathbf{v}^t)'\mathbf{B}^{tr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{e}^{sq}$$

$$=\sum_{t\neq r,z\neq r,q}(\mathbf{v}^t)'(\mathbf{B}^{tz} - \mathbf{C}^{(r)tz})\mathbf{A}^{zr}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{e}^{sq}$$

$$+\sum_{t\neq r,z\neq r,q}(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{e}^{sq}$$

Recall from equation (7) that $\mathbf{e}^{sq} = \sum_{k,l}\mathbf{A}^{sq}\mathbf{B}^{qk}\mathbf{y}^{kl} + \mathbf{y}^{sq}$. The elements $\sum_{t\neq r,z\neq r,q}(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{y}^{sq}$ are thus part of the second term in (22) and belong to the FDC in KWW. At the same time, however, they are included in our FVA. Using $\mathbf{L}^{ss} \leq \mathbf{B}^{ss}$ yields that $\sum_{t\neq r,z\neq r,q}(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{y}^{sq}$ is included in $\sum_{t\neq r,z\neq r,q}(\mathbf{v}^t)'\mathbf{C}^{(r)tz}\mathbf{A}^{zr}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{ss}\mathbf{y}^{sq}$. In turn, this is included in part (*iv*) of (10), which can be seen by taking $p = s$. Hence, part of the FDC in KWW belongs to FVAiT.

[9] It follows immediately from equation (A5) in Appendix 3 that $\mathbf{L}^{ss} \leq \mathbf{B}^{ss}$. The entire first term in (21) is thus included in part (*iv*) of (10).

# 5 Conclusions

The decomposition of the factor content of exports is a useful tool when studying international trade. It has been applied to topics such as the economic impact of trade, comparative advantage, bilateral trade balances, trade costs, effective real exchange rates, global value chains, and vertical specialisation (Arto et al., 2015; Foster-McGregor and Stehrer, 2013; Hummels et al., 2001; Johnson, 2014, 2018; Kee and Tang, 2016; Koopman et al., 2014; Timmer et al., 2014). In this report, we have developed a new accounting framework to completely decompose the bilateral gross exports as measured at the border. A single expression covers the main measures that have been proposed in the literature for assessing the factor content of trade and the fragmentation of global value chains. These measures are: domestic and foreign value added in trade, trade in value added (or value added exports), double counted value added terms, and the VAX ratio. The framework can also easily be linked to other measures such as vertical specialisation (Hummels et al., 2001) and the domestic content in intermediate exports that finally return home (Daudin et al., 2011).

This method decomposes *bilateral* trade measured at the border (e.g. US exports of iron to China), taking into account the different participants in the global value chain and tracing the links from the country and industry in which value added has been generated to the country whose final demand absorbs it. The method takes full account of the exporting and importing countries and industries that ultimately enable the transfers of value added.

In this framework all components of the decomposition of the gross exports depend on the demand for final goods. In this sense, this approach is consistent with the foundations of Leontief's "demand-driven" input-output model. This decomposition of bilateral exports might thus be used to assess the impact in the factor content of trade derived from a shock in the final demands (ceteris paribus).

The accounting approach presented in this report allows showing simultaneously the information on the exporting country, the importing country, the country in which the value added is generated and the country which absorbs the value added.

The full decomposition of gross exports is only meaningful in monetary terms (i.e. in value added). Yet, this approach for the value added content of trade can also be applied to study supply chains in relation to the use of other production factors. Examples are employment, resources (energy, land, water, materials), or the generation of pollution (e.g. $CO_2$ emissions). For instance, the expression for the foreign value added in intermediate exports can be used to quantify the employment in the agriculture sector of the US that is embodied in the exports of textiles from Mexico to Canada, that end up in the Japanese consumption of clothes produced in the US.

This framework paves the way to develop new statistics and indicators with the ultimate aim of providing policy-relevant evidence-based information, as shown by three recent European Commission's reports (Arto et al. 2018a, 2018b, 2018c) that used Trade-SCAN[10] 1.1, a new software tool based on the methodology proposed in this report. These results are necessary to better understand issues such as global value chains, vertical specialisation, bilateral trade balances and overlapping tariffs.

[10] Román et al. (2019)

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## List of figures

## Appendix 1

In this appendix we prove that $\sum_{r,i} v_i^r b_{ij}^{rs} = 1$. $\sum_{r,i} v_i^r b_{ij}^{rs}$ is the $j$th element of the $1 \times m$ vector $\sum_r (\mathbf{v}^r)'\mathbf{B}^{rs}$. Note that $\mathbf{v}' = \mathbf{u}'_{(nm)}(\mathbf{I}-\mathbf{A})$ and $\mathbf{B} = (\mathbf{I}-\mathbf{A})^{-1}$. Hence

$$\mathbf{v}'\mathbf{B} = (\,\sum_r (\mathbf{v}^r)'\mathbf{B}^{r1} \;\cdots\; \sum_r (\mathbf{v}^r)'\mathbf{B}^{rs} \;\cdots\; \sum_r (\mathbf{v}^r)'\mathbf{B}^{rn}\,)$$

$$= \mathbf{u}'_{(nm)}(\mathbf{I}-\mathbf{A})\mathbf{B} = \mathbf{u}'_{(nm)}$$

## Appendix 2

In this appendix we present the components of the Leontief inverse matrix of the multi-regional input-output model in its partitioned form. These components will be derived on the basis of the three Leontief inverse matrices that have been defined in the main text of this report: the Leontief inverse of the multi-regional input-output model ($\mathbf{B}$), the Leontief inverse of the single country model ($\mathbf{L}$), and the Leontief inverse matrix resulting from eliminating from the multi-regional matrix the rows and columns corresponding to country *r* ($\mathbf{C}^{(r)}$).

Following the properties of the inverse of the partitioned matrix, the elements of $\mathbf{B}$ are given as follows:

(A1) $$\mathbf{B}^{rr} = ((\mathbf{I} - \mathbf{A}^{rr}) - \textstyle\sum_{s\neq r,p\neq r} \mathbf{A}^{rs}\mathbf{C}^{(r)sp}\mathbf{A}^{pr})^{-1}$$

(A2) $$\mathbf{B}^{ss} = ((\mathbf{I} - \mathbf{A}^{ss}) - \textstyle\sum_{r\neq s,p\neq s} \mathbf{A}^{sr}\mathbf{C}^{(s)rp}\mathbf{A}^{ps})^{-1}$$

(A3) $$\mathbf{B}^{rs} = \textstyle\sum_{p\neq r} \mathbf{L}^{rr}\mathbf{A}^{rp}\mathbf{B}^{ps} = \sum_{p\neq r} \mathbf{B}^{rr}\mathbf{A}^{rp}\mathbf{C}^{(r)ps}$$

(A4) $$\mathbf{B}^{sr} = \textstyle\sum_{p\neq r} \mathbf{B}^{sp}\mathbf{A}^{pr}\mathbf{L}^{rr} = \sum_{p\neq r} \mathbf{C}^{(r)sp}\mathbf{A}^{pr}\mathbf{B}^{rr}$$

Denoting $\sum_{s\neq r,p\neq r} \mathbf{A}^{rs}\mathbf{C}^{(r)sp}\mathbf{A}^{pr} = \mathbf{K}^{rr}$, we have $\mathbf{B}^{rr} = [(\mathbf{I} - \mathbf{A}^{rr}) - \mathbf{K}^{rr}]^{-1} = \mathbf{L}^{rr}(\mathbf{I} - \mathbf{K}^{rr}\mathbf{L}^{rr})^{-1} = \mathbf{L}^{rr} + \mathbf{L}^{rr}\mathbf{K}^{rr}\mathbf{L}^{rr} + (\mathbf{L}^{rr}\mathbf{K}^{rr})^2\mathbf{L}^{rr} + \cdots$. The difference between $\mathbf{B}^{rr}$ and $\mathbf{L}^{rr}$ is given by $\mathbf{L}^{rr}\mathbf{K}^{rr}\mathbf{L}^{rr} + (\mathbf{L}^{rr}\mathbf{K}^{rr})^2\mathbf{L}^{rr} + \cdots$, which indicates the value chains (starting from and ending in country *r*) that leave country *r* once, twice, etcetera. For the interpretation of the matrix $\mathbf{K}^{rr} = \sum_{s\neq r,p\neq r} \mathbf{A}^{rs}\mathbf{C}^{(r)sp}\mathbf{A}^{pr}$, assume a final demand increase in country *r*. First, this raises the output through domestic chains (reflected by $\mathbf{L}^{rr}$), for which imported inputs (and thus production abroad) is required, reflected by the term $\sum_{p\neq r} \mathbf{A}^{pr}\,\mathbf{L}^{rr}$. Next, this extra production generates more production abroad through strictly foreign chains, which is reflected by $\sum_{s\neq r,p\neq r} \mathbf{C}^{(r)sp}\mathbf{A}^{pr}\,\mathbf{L}^{rr}$. This extra production abroad requires inputs from country *r* ($\sum_{s\neq r,p\neq r} \mathbf{A}^{rs}\mathbf{C}^{(r)sp}\mathbf{A}^{pr}\,\mathbf{L}^{rr}$), which requires more production in *r* through domestic value chains ($\sum_{s\neq r,p\neq r} \mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{C}^{(r)sp}\mathbf{A}^{pr}\,\mathbf{L}^{rr}$), after which another round starts.

## Appendix 3

In this appendix we prove that

$$\sum_{p,q\neq r}(\mathbf{v}^r)'\mathbf{B}^{rp}\mathbf{y}^{pq}=\sum_{s\neq r,p,q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}+\sum_{s\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{y}^{rs}$$

Consider $\mathbf{Q}=(\mathbf{I}-\mathbf{A}^{rr})-\sum_{s\neq r,p\neq r}\mathbf{A}^{rs}\mathbf{C}^{(r)sp}\mathbf{A}^{pr}$. Then, it can be written that: $\mathbf{Q}=(\mathbf{I}-\mathbf{A}^{rr})-\sum_{s\neq r}\mathbf{A}^{rs}\sum_{p\neq r}\mathbf{C}^{(r)sp}\mathbf{A}^{pr}=(\mathbf{I}-\mathbf{A}^{rr})-\sum_{s\neq r}\mathbf{A}^{rs}\sum_{p\neq r}\mathbf{C}^{(r)sp}\mathbf{A}^{pr}\mathbf{B}^{rr}(\mathbf{B}^{rr})^{-1}=$ $(\mathbf{I}-\mathbf{A}^{rr})-\sum_{s\neq r}\mathbf{A}^{rs}\mathbf{B}^{sr}(\mathbf{B}^{rr})^{-1}$, where the last step follows from expression (A4) for $\mathbf{B}^{sr}$. This implies $\mathbf{L}^{rr}\mathbf{Q}\mathbf{B}^{rr}=\mathbf{B}^{rr}-\sum_{s\neq r}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sr}$. On the other hand, it follows directly from expression (A1) that $\mathbf{Q}=(\mathbf{B}^{rr})^{-1}$ and $\mathbf{L}^{rr}\mathbf{Q}\mathbf{B}^{rr}=\mathbf{L}^{rr}$. We thus have obtained that $\mathbf{L}^{rr}=\mathbf{B}^{rr}-\sum_{s\neq r}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sr}$ or

(A5) $$\mathbf{B}^{rr}=\mathbf{L}^{rr}+\sum_{s\neq r}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sr}$$

This implies

(A6) $$\sum_{q\neq r}(\mathbf{v}^r)'\mathbf{B}^{rr}\mathbf{y}^{rq}=\sum_{s\neq r,q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sr}\mathbf{y}^{rq}+\sum_{q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{y}^{rq}$$

(A3) states that $\mathbf{B}^{rp}=\sum_{s\neq r}\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}$. Next add $\sum_{p\neq r,q\neq r}(\mathbf{v}^r)'\mathbf{B}^{rp}\mathbf{y}^{pq}$ on the left-hand side of (A6) and add $\sum_{p\neq r,s\neq r,q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$ on the right-hand side. We then have

$$\sum_{q\neq r}(\mathbf{v}^r)'\mathbf{B}^{rr}\mathbf{y}^{rq}+\sum_{p\neq r,q\neq r}(\mathbf{v}^r)'\mathbf{B}^{rp}\mathbf{y}^{pq}=\sum_{q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{y}^{rq}$$

$$+\sum_{s\neq r,q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sr}\mathbf{y}^{rq}+\sum_{p\neq r,s\neq r,q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$$

Collecting terms yields

$$\sum_{p,q\neq r}(\mathbf{v}^r)'\mathbf{B}^{rp}\mathbf{y}^{pq}=\sum_{q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{y}^{rq}+\sum_{p,s\neq r,q\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sp}\mathbf{y}^{pq}$$

which completes the proof.

## Appendix 4

In this appendix we first prove that $\sum_{t\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rt}\,\mathbf{B}^{tr}\mathbf{y}^{rs} = (\mathbf{v}^r)'(\mathbf{B}^{rr} - \mathbf{L}^{rr})\mathbf{y}^{rs}$.

Consider $\mathbf{R} = \sum_{t\neq r,p\neq r}\mathbf{A}^{rt}\mathbf{C}^{(r)tp}\mathbf{A}^{pr}$. On the one hand it obviously equals $\mathbf{R} = \sum_{t\neq r,p\neq r}\mathbf{A}^{rt}\mathbf{C}^{(r)tp}\mathbf{A}^{pr}\,\mathbf{B}^{rr}(\mathbf{B}^{rr})^{-1}$. Using expression (A4) gives $\sum_{p\neq r}\mathbf{C}^{(r)tp}\mathbf{A}^{pr}\mathbf{B}^{rr} = \mathbf{B}^{tr}$, which then yields $\mathbf{R} = \sum_{t\neq r}\mathbf{A}^{rt}\,\mathbf{B}^{tr}(\mathbf{B}^{rr})^{-1}$. Pre-multiplication with $\mathbf{L}^{rr}$ and post-multiplication with $\mathbf{B}^{rr}$ implies $\mathbf{L}^{rr}\mathbf{R}\mathbf{B}^{rr} = \sum_{t\neq r}\mathbf{L}^{rr}\mathbf{A}^{rt}\,\mathbf{B}^{tr}$.

On the other hand $\mathbf{R} = (\mathbf{I} - \mathbf{A}^{rr}) - \left[(\mathbf{I} - \mathbf{A}^{rr}) - \sum_{t\neq r,p\neq r}\mathbf{A}^{rt}\mathbf{C}^{(r)tp}\mathbf{A}^{pr}\right] = (\mathbf{I} - \mathbf{A}^{rr}) - (\mathbf{B}^{rr})^{-1}$, where we have used (A1). Therefore $\mathbf{L}^{rr}\mathbf{R}\mathbf{B}^{rr} = \mathbf{B}^{rr} - \mathbf{L}^{rr}$.

We now have $\mathbf{B}^{rr} - \mathbf{L}^{rr} = \sum_{t\neq r}\mathbf{L}^{rr}\mathbf{A}^{rt}\,\mathbf{B}^{tr}$, so that $(\mathbf{v}^r)'(\mathbf{B}^{rr} - \mathbf{L}^{rr})\mathbf{y}^{rs} = \sum_{t\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rt}\,\mathbf{B}^{tr}\mathbf{y}^{rs}$.

The conclusion that follows from this is that *v* $[(\mathbf{v}^r)'(\mathbf{B}^{rr} - \mathbf{L}^{rr})\mathbf{y}^{rs} = \sum_{t\neq r}(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rt}\,\mathbf{B}^{tr}\mathbf{y}^{rs}]$ includes the term $(\mathbf{v}^r)'\mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{B}^{sr}\mathbf{y}^{rs}$, namely when $t = s$. This term is also included in *ii*, namely for $p = r$ and $q = s$. This means that *v* and *ii* partly overlap.

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